A couple of facts about centralizers & normalizers:

① Recall If G is a group and $S \subseteq G$, then $\{e\} \leq Z(G) \leq C_G(S) \leq N_G(S) \leq G$.

identity

② If H is a subgroup of G, then

$$H \triangleleft N_G(H) < G$$

largest subgroup in G such that H is normal in it.

Conjugacy Class of $\alpha \in G = \{g\alpha g^{-1} : g \in G\} = cl(\alpha)$

If $\alpha \in Z(G)$, then $cl(\alpha) = \{\alpha\}$.

Example: If G acts on a set X, with $x \in X$, Gx = orbit of x

G_x = isotropy subgroup at x.

Notice if $y = hx \in Gx$, then

$g \in G_y \iff gy = y \iff ghx = hx \iff h^{-1}ghx = x$

$\iff h^{-1}gh \in G_x \iff g \in hG_xh^{-1}$

$\therefore \boxed{G_{hx} = hG_xh^{-1}}$ Along an orbit, the isotropy subgroups are conjugate to each other.

✱ If $H < G$, and if $\tau \in G$, then $\tau H \tau^{-1}$ is a subgroup of G.

- If H is a finite subgroup of G and $g \in G$, then $|gHg^{-1}| = |H|$.

<u>Proof:</u> Define $F_g : H \to gHg^{-1}$ be given by $F_g(h) = ghg^{-1}$. Then F_g is a homomorphism, since $\forall h_1, h_2 \in H$, $F_g(h_1h_2) = g h_1 h_2 g^{-1}$

$$= g h_1 g^{-1} g h_2 g^{-1} = F_g(h_1) F_g(h_2).$$

F_g is onto, because $\forall\, ghg^{-1} \in H$, $\exists h \in H$ s.t.

$$F_g(h) = ghg^{-1}. \checkmark$$

Also, $\mathrm{Ker}(F_g) = \{\beta \in H : F_g(\beta) = e\}$

$$= \{\beta \in H : g\beta g^{-1} = e\}$$

$$\Updownarrow$$

$$\beta g^{-1} = g^{-1}e$$

$$\Updownarrow$$

$$\beta = g^{-1}eg = e$$

$$= \{\beta \in H : \beta = e\}$$

$$= \{e\}.$$

$\therefore F_g$ is 1-1.

<u>Lemma</u> $|cl(a)| = \underbrace{[G : C_G(a)]}_{\text{Index of the subgroup } C_G(a) \text{ inside } G}$ for all $a \in$ group G.

$$= |G / C_G(a)| \overset{\text{if } G \text{ finite}}{=} \frac{|G|}{|C_G(a)|}$$

<u>Cor</u> $|cl(a)|$ is a factor of $o(G)$, if G is finite.

Fundamental Theorem for Finitely Generated Abelian Groups

(FTFGAG): Let G be finitely generated abelian group ($\exists\ a_1, \dots, a_k \in G$ s.t. every element of G can be written $a_1^{m_1} a_2^{m_2} a_3^{m_3} \cdots a_k^{m_k}$ for some integers $m_1, \dots, m_k$). Then G is isomorphic to

$$\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_s^{r_s}} \times \mathbb{Z}^q$$

for some $q \geq 0$, primes $p_1, \dots, p_s$ (not nec. distinct), positive integers $r_1, r_2, \dots, r_s$.

Aside: Since $\mathbb{Z}_a \times \mathbb{Z}_b \cong \mathbb{Z}_{ab}$ if $(a,b)=1$.

sometimes people write the above as

$$\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \mathbb{Z}_{n_c} \times \mathbb{Z}^q$$

where $n_1 | n_2 | n_3 \cdots | n_c$ $\qquad c \in \mathbb{Z}_{\geq 0}$.

Corollary: If G is a finite abelian group such that $p \,|\, |G|$ for some prime p, then there exists an element of order p in G.

Pf: By FTFGAG, with given $G \cong \mathbb{Z}_{p^k} \times H$, $k \geq 1$, H everything else.

and $(p^{k-1}, 0)$ is an element of order p in $\mathbb{Z}_{p^k} \times H$. $\blacksquare$

Thm Class equation:

Let G be a finite group, and let x_j be a specific element of the j^{th} nontrivial conjugacy class, where j runs through the different conjugacy classes,

Then $$|G| = |Z(G)| + \sum_j |C\ell(x_j)|$$
$$= |Z(G)| + \sum_j [G : C_G(x_j)]$$